Journal of Applied Mathematics and Stochastic Analysis, 11:2 (1998), 107-114.

ON THE APPROXIMATION OF AN INTEGRAL BY A SUM OF RANDOM VARIABLES

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(Received October, 1996; Revised September, 1997)

We approximate the integral of a smooth function on $[0,1]$, where values are only known at n random points (i.e., a random sample from the uniform-$(0,1)$ distribution), and at 0 and 1. Our approximations are based on the trapezoidal rule and Simpson's rule (generalized to the non-equidistant case), respectively. In the first case, we obtain an n^2-rate of convergence with a degenerate limiting distribution; in the second case, the rate of con-vergence is as fast as $n^{3^1/2}$, whereas the limiting distribution is Gaussian then.

Key words: Numerical Integration, Order Statistics, Spacings.
AMS subject classifications: 60F05, 62G30, 65D30.

1. Introduction and Main Results

Suppose we (can) only observe the values of a smooth function $f:[0,1]\to\mathbb{R}$ at the points $U_0, U_1, \ldots, U_n, U_{n+1}$, where $U_1, U_2, \ldots, U_n$ are the order statistics ($U_1 \le U_2 \le \ldots \le U_n$) of n independent uniformly-$(0,1)$ distributed random variables and $U_0{:} = 0$, $U_{n+1}{:} = 1$. It is our aim to estimate the integral

$$I{:} = \int_0^1 f(x)dx \tag{1}$$

from these observations, i.e., by only using $(U_i, f(U_i))$, $i = 0, 1, \ldots, n+1$. The first

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[1]Research partially supported by European Union HCM grant ERB CHRX-CT 940693.

estimator we will employ is constructed by using the 'trapezoidal rule' on each subinterval $[U_{i-1}, U_i]$, $i = 1, \ldots, n+1$. This rule approximates an integral $\int_a^b g(x)dx$ simply by $\frac{1}{2}(b-a)(g(a)+g(b))$ and it can easily be shown (see, e.g., Isaacson and Keller [2], p. 304) that

$$\frac{1}{2}(b-a)(g(a)+g(b)) - \int_a^b g(x)dx = \frac{1}{12}(b-a)^3 g''(\eta), \tag{2}$$

where $\eta \in (a,b)$. Writing $D_i = U_i - U_{i-1}$, $i = 1, \ldots, n+1$, for the spacings of the U_i's, our estimator of I becomes

$$I_n := \sum_{i=1}^{n+1} \frac{1}{2} D_i (f(U_{i-1}) + f(U_i)). \tag{3}$$

Using (2), we will prove the following limiting result for the standardized difference of I_n and I:

Theorem 1: *If* $|f'''|$ *is bounded, then*

$$n^2(I_n - I) \xrightarrow{P} \frac{1}{2}(f'(1) - f'(0)), \quad \text{as } n \to \infty. \tag{4}$$

A much better and probabilistically more interesting estimator is obtained by applying a 3-points formula, i.e., for a given $c \in (a,b)$, we approximate $\int_a^b g(x)dx$ by $w_1 g(a) + w_2 g(c) + w_3 g(b)$ in such a way that the approximation error is zero in the case g is a polynomial of second degree. If the 3 points are equidistant, this approximation is known as Simpson's rule. It is not hard to show that

$$w_1 = \frac{1}{6}(b-a)\left(2 - \frac{b-c}{c-a}\right),\ w_2 = \frac{1}{6}\frac{(b-a)^3}{(c-a)(b-c)},\ w_3 = \frac{1}{6}(b-a)\left(2 - \frac{c-a}{b-c}\right), \tag{5}$$

and it follows (see again Isaacson and Keller [2], p. 304) that

$$w_1 g(a) + w_2 g(c) + w_3 g(b) - \int_a^b g(x)dx = -\frac{1}{6}\int_a^b (x-a)(x-c)(x-b) g^{(3)}(\eta) dx, \tag{6}$$

where $\eta = \eta(x) \in (a,b)$. Hence, our estimator of I in (1), again denoted by I_n, becomes

$$\begin{aligned} I_n = \sum_{i=1}^{\frac{n+1}{2}} \frac{1}{6}(D_{2i-1} + D_{2i})\{(2 - \frac{D_{2i}}{D_{2i-1}}) f(U_{2i-2}) \\ + \frac{(D_{2i-1}+D_{2i})^2}{D_{2i-1}D_{2i}} f(U_{2i-1}) + (2 - \frac{D_{2i-1}}{D_{2i}}) f(U_{2i})\}, \end{aligned} \tag{7}$$

where, for convenience, n is taken to be odd. Formula (6) will be used to prove our main result:

Theorem 2: *Let n be odd. If* $|f^{(5)}|$ *is bounded, then*

$$n^{3\frac{1}{2}}(I_n - I) \xrightarrow[d]{} \sqrt{\frac{35}{3}\int_0^1 (f^{(3)}(x))^2 dx}\ Z, \quad \text{as } n \to \infty, \tag{8}$$

where Z is a standard normal random variable.

Remark 1: The present techniques can be easily adapted to cover the situation where the U_i's are the order statistics of n independent random variables with common distribution function G (on $(0,1)$) having a smooth density g. The adaptation is based on the quantile transform, transforming a uniform random variable V into a random variable $G^{-1}(V)$ with distribution function G. In this case, under regularity conditions on g, we obtain that the weak limit in Theorem 1 becomes $\frac{1}{2}\int_0^1 (f''(x)/g^2(x))dx$ instead of $\frac{1}{2}\int_0^1 f''(x)dx = \frac{1}{2}(f'(1)-f'(0))$. In Theorem 2, the limiting random variable is again centered normal but now the standard deviation becomes

$$\sqrt{\frac{35}{3}\int_0^1 \frac{(f^{(3)}(x))^2}{g^7(x)}dx}.$$

On the other hand, the uniform distribution seems very relevant because of the following. Since $\int_0^1 f(x)dx$ can be considered as the mean 'output', given that the x-values are 'equally important', it seems desirable to estimate $\int_0^1 f(x)g(x)dx = \int_0^1 f(G^{-1}(y))dy$ in the case the random variables are distributed according to G. But if G is *known*, we can replace the pairs $(U_i, f(U_i))$ (just below (1)), with U_i's being the order statistics from G, by $(G(U_i), f(U_i)) = (G(U_i), f(G^{-1}(G(U_i))))$. This brings us back to the 'uniform distribution setup' with f replaced by $f \circ G^{-1}$, but that is just the function whose integral we wanted to estimate as argued above!

This idea leads to possible ways of applying the results. Suppose U_i represents some uncontrollable physical random quantity, like temperature, humidity or light intensity with a known distribution function G having density g. Suppose also that we can measure f (the output or yield) only at the U_i and that we are interested in the mean output $I_g = \int_0^1 f(x)g(x)dx$. Then one can use our theorems to obtain rapidly converging estimators of I_g. In particular, when measuring the f-values is hard or expensive, one can get good estimators based on a few observations.

Also note that for the trapezoidal rule in Theorem 1 and f'' being constant, the uniform distribution is optimal, since $\int_0^1 g^{-2}(x)dx \geq \int_0^1 1dx = 1$. (This can be easily seen by using Jensen's inequality:

$$\int_0^1 \frac{1}{g^2(x)}dx = \int_0^1 \frac{1}{g^3(x)}g(x)dx = \mathbb{E}\frac{1}{g^3(X)} \geq \left(\mathbb{E}\frac{1}{g(X)}\right)^3$$

$$= \left(\int_0^1 \frac{1}{g(x)}g(x)dx\right)^3 = 1,$$

where X is a random variable with density g.) A similar remark applies to Theorem 2 with $f^{(3)}$ being constant.

Remark 2: There are various other ways to extend our results, which we will not pursue here, e.g., applying m-points formulas for $m > 3$ (Simpson's rule is 'by far the most frequently used in obtaining approximate integrals', Davis and Rabinowitz [1], p. 45), combining trapezoidal rules to eliminate the bias $\frac{1}{2}(f'(1)-f'(0))$, proving a 'second order' limit result for $n^2(I_n-I)-\frac{1}{2}(f'(1)-f'(0))$ in Theorem 1, or treating the case n 'even' in Theorem 2. We are not pursuing these extensions because we believe they are not very interesting and/or they do not give good results.

Remark 3: We briefly compare our results with the deterministic, equidistant case, i.e., $U_i = \frac{i}{n+1}$, $i = 0, 1, \ldots, n+1$. It is well-known that the limit in Theorem 1 is $\frac{1}{12}(f'(1)-f'(0))$ in that case, which means that we loose a factor of 6 by having random U_i's. (Essentially, this 6 is coming from the third moment of a standard exponential random variable.) From Theorem 2, it is well-known that in the equidistant case (Simpson's rule), the rate is n^4. So, there our loss is of order $n^{1/2}$. Nevertheless, from statistical point of view, $n^{3\frac{1}{2}}$ is a remarkably fast rate of convergence.

2. Proofs

The following well-known lemma will be used frequently; it can be found in, e.g., Shorack and Wellner [3], p. 721.

Lemma 1: *Let $E_1, \ldots, E_{n+1}$ be independent exponential random variables with mean 1 and S_{n+1} be their sum. With D_i, $i = 1, \ldots, n+1$, as before, we have*

$$(D_1, \ldots, D_{n+1}) \stackrel{d}{=} \left(\frac{E_1}{S_{n+1}}, \ldots, \frac{E_{n+1}}{S_{n+1}} \right).$$

Proof of Theorem 1: Using (3), (1) and (2) we see that

$$n^2(I_n - I) = \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 f''(\tilde{U}_i)$$

for some $\tilde{U}_i \in (U_{i-1}, U_i)$, and hence,

$$n^2(I_n - I) = \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 f''\Big(\frac{i}{n+1}\Big) + \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 \Big(\tilde{U}_i - \frac{i}{n+1}\Big) f'''(\tilde{\tilde{U}}_i), \tag{9}$$

with $\tilde{\tilde{U}}_i$ between $\tilde{U}_i$ and $\frac{i}{n+1}$. From the boundedness of $|f'''|$ (by M, say) and the weak convergence (to a Brownian bridge) of the uniform quantile process (see, e.g., Shorack and Wellner [3]), it is readily seen that

$$\begin{aligned} &\left| \frac{n^2}{12} \sum_{i=1}^{n+1} D_i^3 \Big(\tilde{U}_i - \frac{i}{n+1}\Big) f'''(\tilde{\tilde{U}}_i) \right| \\ &\leq \frac{n^2}{12} M \sup_{i \in \{1, \ldots, n+1\}} \left| \tilde{U}_i - \frac{i}{n+1} \right| \sum_{i=1}^{n+1} D_i^3 = O_p(n^{1\frac{1}{2}}) \sum_{i=1}^{n+1} D_i^3. \end{aligned} \tag{10}$$

But

$$\sum_{i=1}^{n+1} D_i^3 \stackrel{d}{=} \frac{1}{S_{n+1}^3} \sum_{i=1}^{n+1} E_i^3, \tag{11}$$

by Lemma 1, and by two applications of the weak law of large numbers, this last expression is $O_p(n^{-2})$. Combining this with (10) and (11) yields that the second term on the right in (9) converges to zero in probability. Hence, it remains to consider the first term

$$\frac{n^2}{12}\sum_{i=1}^{n+1} D_i^3 f''\Big(\frac{i}{n+1}\Big) \stackrel{d}{=} \frac{1}{12n}\left(\frac{n}{S_{n+1}}\right)^3 \sum_{i=1}^{n+1} f''\Big(\frac{i}{n+1}\Big)E_i^3,$$

or, since $(n/S_{n+1})^3 \xrightarrow{P} 1$,

$$\frac{1}{12n}\sum_{i=1}^{n+1} f''\Big(\frac{i}{n+1}\Big)E_i^3.$$

By Chebysev's inequality, it follows that

$$\frac{1}{12n}\sum_{i=1}^{n+1} f''\Big(\frac{i}{n+1}\Big)E_i^3 - \frac{1}{2n}\sum_{i=1}^{n+1} f''\Big(\frac{i}{n+1}\Big) \xrightarrow{P} 0.$$

The proof is complete by noting that

$$\frac{1}{2n}\sum_{i=1}^{n+1} f''\Big(\frac{i}{n+1}\Big) \to \frac{1}{2}\int_0^1 f''(x)dx = \frac{1}{2}(f'(1)-f'(0)). \qquad \square$$

The proof of Theorem 2 is heavily based on the following two lemmas.

Lemma 2: *Let $E_1,\ldots,E_{n+1}$, n odd, be independent exponential random variables with mean 1. Write*

$$X_i = (E_{2i-1}+E_{2i})^3(E_{2i}-E_{2i-1}),\ \ i=1,2,\ldots,\frac{n+1}{2},$$

$$Y_i = X_i\sum_{j=1}^{2i-2}(E_j-1), \qquad i=2,3,\ldots,\frac{n+1}{2}.$$

Then,

$$\mathbb{E}X_i = 0,\ \ \operatorname{Var}X_i = 120960,\ \ \mathbb{E}Y_i = 0,\ \operatorname{Var}Y_i = 120960(2i-2),$$

$$\operatorname{Cov}(Y_i,Y_k) = 0, \text{ for } i \neq k.$$

Proof: By symmetry, we see that $\mathbb{E}X_i = 0$; a straightforward computation yields $\operatorname{Var}X_i = \mathbb{E}X_i^2 = 120960$. For the Y_i's we have

$$\mathbb{E}Y_i = \mathbb{E}X_i\mathbb{E}\sum_{j=1}^{2i-2}(E_j-1) = 0,$$

$$\operatorname{Var}Y_i = \mathbb{E}Y_i^2 = \mathbb{E}X_i^2\mathbb{E}\left(\sum_{j=1}^{2i-2}(E_j-1)\right)^2$$

$$= \operatorname{Var}X_i \operatorname{Var}\left(\sum_{j=1}^{2i-2}E_j\right) = 120960(2i-2),$$

and for $i<k$,

$$\operatorname{Cov}(Y_i,Y_k) = \mathbb{E}Y_iY_k$$

$$= \mathbb{E}X_k\left(\sum_{j=1}^{2k-2}(E_j-1)\right)X_i\left(\sum_{j=1}^{2i-2}(E_j-1)\right)$$

$$= \mathbb{E}X_k\,\mathbb{E}\left(\sum_{j=1}^{2k-2}(E_j-1)\right)X_i\left(\sum_{j=1}^{2i-2}(E_j-1)\right)$$

$$= 0. \qquad \square$$

Lemma 3: *Under the conditions of Theorem 2, we have, as $n \to \infty$,*

$$\left| n^{3\frac{1}{2}}(I_n - I) - \frac{n^{3\frac{1}{2}}}{72} \sum_{i=1}^{\frac{n+1}{2}} f^{(3)}\Big(\frac{2i-2}{n+1}\Big) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1}) \right| = o_p(1).$$

Proof: By (7), (1) and (6) we have

$$n^{3\frac{1}{2}}(I_n - I) = -\frac{n^{3\frac{1}{2}}}{6} \sum_{i=1}^{\frac{n+1}{2}} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2})(x - U_{2i-1})(x - U_{2i}) f^{(3)}(\widetilde{U}_{2i}) dx, \qquad (12)$$

for some $\widetilde{U}_{2i} = \widetilde{U}_{2i}(x) \in (U_{2i-2}, U_{2i})$ and hence for some $\widetilde{\widetilde{U}}_{2i} = \widetilde{\widetilde{U}}_{2i}(x) \in (U_{2i-2}, \widetilde{U}_{2i})$, the right-hand side of (12) is equal to

$$-\frac{n^{3\frac{1}{2}}}{6} \sum_{i=1}^{\frac{n+1}{2}} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2})(x - U_{2i-1})(x - U_{2i})$$

$$\times (f^{(3)}(U_{2i-2}) + (\widetilde{U}_{2i} - U_{2i-2}) f^{(4)}(\widetilde{\widetilde{U}}_{2i})) dx$$

$$= \frac{n^{3\frac{1}{2}}}{72} \sum_{i=1}^{\frac{n+1}{2}} f^{(3)}(U_{2i-2})(D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1})$$

$$-\frac{n^{3\frac{1}{2}}}{6} \sum_{i=1}^{\frac{n+1}{2}} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2})(x - U_{2i-1})(x - U_{2i})\, (\widetilde{U}_{2i} - U_{2i-2}) f^{(4)}(\widetilde{\widetilde{U}}_{2i}) dx.$$

Let M be a bound on $|f^{(5)}|$ and all lower order derivatives of f. Then the absolute value of this last term is bounded from above by

$$M \frac{n^{3\frac{1}{2}}}{6} \sum_{i=1}^{\frac{n+1}{2}} \int_{U_{2i-2}}^{U_{2i}} (x - U_{2i-2}) \,|\, x - U_{2i-1} \,|\, (U_{2i} - x)(\widetilde{U}_{2i} - U_{2i-2}) dx$$

$$\leq \frac{M}{6} n^{3\frac{1}{2}} \sum_{i=1}^{\frac{n+1}{2}} (D_{2i-1} + D_{2i})^5 \stackrel{d}{=} \frac{M}{6} n^{3\frac{1}{2}} \frac{1}{S_{n+1}^5} \sum_{i=1}^{\frac{n+1}{2}} (E_{2i-1} + E_{2i})^5 = o_p(1),$$

due to Lemma 1 and two applications of the weak law of large numbers.

So, it suffices to show the convergence to zero in probability of

$$\frac{n^{3\frac{1}{2}}}{72} \sum_{i=1}^{\frac{n+1}{2}} \Big(f^{(3)}(U_{2i-2}) - f^{(3)}\Big(\frac{2i-2}{n+1}\Big)\Big) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1})$$

$$= \frac{n^{3\frac{1}{2}}}{72} \sum_{i=2}^{\frac{n+1}{2}} \Big(U_{2i-2} - \frac{2i-2}{n+1}\Big) f^{(4)}\Big(\frac{2i-2}{n+1}\Big) (D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1})$$

$$+\frac{n^{3\frac{1}{2}}}{144}\sum_{i=2}^{\frac{n+1}{2}}\Big(U_{2i-2}-\frac{2i-2}{n+1}\Big)^2 f^{(5)}(\bar{U}_{2i-2})(D_{2i-1}+D_{2i})^3(D_{2i}-D_{2i-1})$$

$$=:T_{1,n}+T_{2,n},$$

for some $\bar{U}_{2i-2}$ between U_{2i-2} and $\frac{2i-2}{n+1}$. By the weak convergence of the uniform quantile process,

$$|T_{2,n}| \le \frac{n^{3\frac{1}{2}}}{144}M \sup_{i\in\{2,3,\dots,\frac{n+1}{2}\}}\Big(U_{2i-2}-\frac{2i-2}{n+1}\Big)^2\sum_{i=2}^{\frac{n+1}{2}}(D_{2i-1}+D_{2i})^4$$

$$=O_p(n^{2\frac{1}{2}})\sum_{i=2}^{\frac{n+1}{2}}(D_{2i-1}+D_{2i})^4.$$

By Lemma 1 and twice the weak law of large numbers, this last expression is easily seen to be $o_p(1)$. Hence, the proof of Lemma 3 is complete if we show $T_{1,n}=o_p(1)$.

From Lemma 1 we obtain

$$T_{1,n}\stackrel{d}{=}\frac{n^{3\frac{1}{2}}}{72}\sum_{i=2}^{\frac{n+1}{2}}\left(\frac{\sum_{j=1}^{2i-2}E_j}{S_{n+1}}-\frac{2i-2}{n+1}\right)f^{(4)}\Big(\frac{2i-2}{n+1}\Big)\Big(\frac{E_{2i-1}+E_{2i}}{S_{n+1}}\Big)^3\Big(\frac{E_{2i}-E_{2i-1}}{S_{n+1}}\Big)$$

$$=\frac{n^{3\frac{1}{2}}}{72}S_{n+1}^{-5}\sum_{i=2}^{\frac{n+1}{2}}\left(\sum_{j=1}^{2i-2}(E_j-1)\right)f^{(4)}\Big(\frac{2i-2}{n+1}\Big)(E_{2i-1}+E_{2i})^3(E_{2i}-E_{2i-1})$$

$$+\frac{n^{3\frac{1}{2}}}{72}S_{n+1}^{-5}\Big(1-\frac{S_{n+1}}{n+1}\Big)\sum_{i=2}^{\frac{n+1}{2}}(2i-2)f^{(4)}\Big(\frac{2i-2}{n+1}\Big)(E_{2i-1}+E_{2i})^3(E_{2i}-E_{2i-1})$$

$$=:T_{3,n}+T_{4,n}.$$

It is immediate from the central limit theorem for $S_{n+1}/(n+1)$ that

$$T_{4,n}=O_p(n^{-2})\sum_{i=2}^{\frac{n+1}{2}}(2i-2)f^{(4)}\Big(\frac{2i-2}{n+1}\Big)X_i,$$

where the X_i's are as in Lemma 2. Now using that lemma in conjunction with Chebysev's inequality, it readily follows that $T_{4,n}=o_p(1)$. Finally, in the notation of Lemma 2,

$$T_{3,n}=\frac{n^{3\frac{1}{2}}}{72}S_{n+1}^{-5}\sum_{i=2}^{\frac{n+1}{2}}f^{(4)}\Big(\frac{2i-2}{n+1}\Big)Y_i=O_p(n^{-1\frac{1}{2}})\sum_{i=2}^{\frac{n+1}{2}}f^{(4)}\Big(\frac{2i-2}{n+1}\Big)Y_i.$$

From Lemma 2, we have

$$\mathbb{E}\sum_{i=2}^{\frac{n+1}{2}}f^{(4)}\Big(\frac{2i-2}{n+1}\Big)Y_i=0,$$

$$\operatorname{Var}\sum_{i=2}^{\frac{n+1}{2}}f^{(4)}\Big(\frac{2i-2}{n+1}\Big)Y_i=\sum_{i=2}^{\frac{n+1}{2}}\Big(f^{(4)}\Big(\frac{2i-2}{n+1}\Big)\Big)^2\operatorname{Var}Y_i=O(n^2).$$

Now, Chebysev's inequality yields $T_{3,n} = o_p(1)$ and hence $T_{1,n} = o_p(1)$. □

Proof of Theorem 2: Given the lemmas, especially Lemma 3, the proof of Theorem 2 is rather easy. If $\int_0^1 (f^{(3)}(x))^2 dx = 0$, then $f^{(3)}(x) = 0$ for all $x \in [0,1]$ and hence trivially $I_n = I$, because f is a polynomial of second degree. Therefore, we assume now $\int_0^1 (f^{(3)}(x))^2 dx > 0$. Using Lemma 1 we have

$$\frac{n^{3\frac{1}{2}}}{72} \sum_{i=1}^{\frac{n+1}{2}} f^{(3)}\Big(\frac{2i-2}{n+1}\Big)(D_{2i-1} + D_{2i})^3 (D_{2i} - D_{2i-1})$$

$$\stackrel{d}{=} \frac{1}{72\sqrt{2}}\Big(\frac{n}{S_{n+1}}\Big)^4 \frac{1}{(n/2)^{1/2}} \sum_{i=1}^{\frac{n+1}{2}} f^{(3)}\Big(\frac{2i-2}{n+1}\Big)(E_{2i-1} + E_{2i})^3 (E_{2i} - E_{2i-1})$$

$$=: \Big(\frac{n}{S_{n+1}}\Big)^4 W_n.$$

By the weak law of large numbers and Lemma 3, it now remains to show Theorem 2 with $n^{3\frac{1}{2}}(I_n - I)$ replaced by W_n. By Lemma 2, we see that $\mathbb{E}W_n = 0$ and

$$\operatorname{Var} W_n = \frac{1}{2(72)^2}\, \frac{2}{n} \sum_{i=1}^{\frac{n+1}{2}} \Big(f^{(3)}\Big(\frac{2i-2}{n+1}\Big)\Big)^2 120960 \to \frac{35}{3}\int_0^1 \Big(f^{(3)}(x)\Big)^2 dx.$$

Now, the Lindeberg central limit theorem applies, because of the boundedness of $| f^{(3)} |$, and it yields the result. □

Acknowledgements

The authors are indebted to A.O.H. Axelsson and B.J.W. Polman for helpful discussions and their useful comments.

References

[1] Davis, P.J. and Rabinowitz, P., *Methods of Numerical Integration*, Academic Press, New York 1975.

[2] Isaacson, I. and Keller, H.B., *Analysis of Numerical Methods*, Wiley, New York 1966.

[3] Shorack, G.R. and Wellner, J.A., *Empirical Processes with Applications to Statistics*, Wiley, New York 1986.